Math Explorations

Take Me to the Limit One More Time

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^{n}$$

Let $\frac{a}{b}$ be positive and less than 1.

Translation:

1. A limit is a value an expression is desperately trying to reach. You will know it when you see it!

2. $\frac{a}{b}$ is a fraction less than one, and of course, b $\neq 0$.

3. $\left(\frac{a}{b}\right)^{n}$ means to raise your fraction to the n^{th} power, first when n=1, then n=2, and forever until n = ∞ (but not really because you will see the pattern before then).

4. $\sum$ means to sum up, in this case all the powers of the fractions you calculated.

Hint: You can use decimals to get to the limit, but it will be hard to see the pattern if you don't write the limit as a fraction. It is a good "attention to detail" exercise to crank the powers of the fractions and the sums out by hand.

Can you do enough examples to make a prediction for the limit of the infinite sum of powers of any fraction less than one?

If you love patterns, read on for an example!

Example: $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$

Step one: $\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$

Step two: $\left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

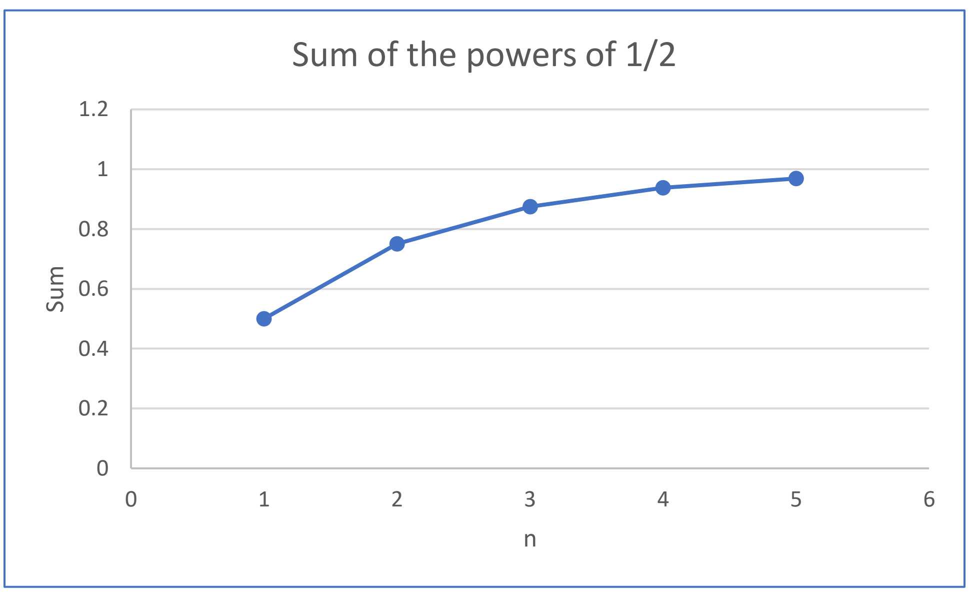
Step three: $\left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ Any idea what this sum is trying to get close to?

Step four: $\left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{3} + \left(\frac{1}{2}\right)^{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Step five: $\left(\frac{1}{2}\right)^{1} + \left(\frac{1}{2}\right)^{2} + \left(\frac{1}{2}\right)^{3} + \left(\frac{1}{2}\right)^{4} + \left(\frac{1}{2}\right)^{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

I am pretty convinced looking at $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$, and $\frac{31}{32}$ that the sum of these fractions is trying really, really hard to get to the number 1.

Here is what the graph of the sums looks like as n moves from 1 to 5.



So, the limit of $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}$ is 1.

Choose other fractions less than 1 and see if there is a pattern which allows you to predict what the sum of the powers of the fraction is trying to get to!

Try starting with unit fractions and keep the denominators small.

I can predict, without doing the math, that $\sum \left(\frac{5}{8}\right)^{n}$, for all values of n = 1 to infinity, is really, really trying to reach the limit of $1\frac{2}{3}$. Can you figure out how I know that?